Rotating black strings in $f(R)$-Maxwell theory

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In general, the field equations of $f(R)$ theory coupled to a matter field are very complicated and hence it is not easy to find exact analytical solutions. However, if one considers traceless energy-momentum tensor for the matter source as well as constant scalar curvature, one can derive some exact analytical solutions from $f(R)$ theory coupled to a matter field. In this paper, by assuming constant curvature scalar, we construct a class of charged rotating black string solutions in $f(R)$-Maxwell theory. We study the physical properties and obtain the conserved quantities of the solutions. The conserved and thermodynamic quantities computed here depend on function $f'(R_0)$ and differ completely from those of Einstein theory in AdS spaces. Besides, unlike Einstein gravity, the entropy does not obey the area law. We also investigate the validity of the first law of thermodynamics as well as the stability analysis in the canonical ensemble, and show that the black string solutions are always thermodynamically stable in $f(R)$-Maxwell theory with constant curvature scalar. Finally, we extend the study to the case where the Ricci scalar is not a constant and in particular $R = R(r)$. In this case, by using the Lagrangian multipliers method, we derive an analytical black string solution from $f(R)$ gravity and reconstructed the function $R(r)$. We find that this class of solutions has an additional logaritmic term in the metric function which incorporates the effect of the $f(R)$ theory in the solutions.

Keywords: modified gravity; string; thermodynamics.

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I. INTRODUCTION

There has been considerable attentions in the past years in modified gravity theories, specially $f(R)$ theory which is one of the encouraging candidates for explaining the current accelerating of the universe expansion [1, 2] (see also [3] for a comprehensive review on $f(R)$ theories). In fact $f(R)$ theories can be regarded as the simplest extension of general relativity. Many $f(R)$ models have passed all the available experimental tests and fit the cosmological data. To prevent a ghost state, $f'(R) > 0$ for $R \geq R_0$ is required [4, 5]. $f''(R) > 0$ for $R \geq R_0$, is needed to avoid the negative mass squared of a scalar-field degree of freedom (tachyon) [3]. $f(R) \to R - 2\Lambda$ for $R \geq R_0$, is required for the presence of the matter era and for consistency with local gravity constraints [3]. It was shown that $f(R)$ theories can be considered as general relativity with an additional scalar field that provide new insight in the two cases of Brans-Dicke theory with $\omega_0 = 0$ and $\omega_0 = -3/2$ [6].

There have been a lot of works in the literature attempting to construct static and stationary black hole solutions in $f(R)$ gravity theories. One may expect that some signatures of black holes in $f(R)$ theories will be in disagreement with the expected physical results of Einstein's gravity. In [7] the authors studied general solutions in $f(R)$ theory using a perturbation approach around the Einstein-Hilbert action. In [8] black hole solutions were found by adding dynamical vector and tensor degrees of freedom to the Einstein-Hilbert action. Also, the transition from neutron stars to a strong scalar- field state in $f(R)$ gravity has been studied in [9]. Physical properties of the matter forming an accretion disk in the spherically symmetric background in $f(R)$ theories were explored in [10]. In Ref. [11] the construction of traversable wormhole geometries was discussed in $f(R)$ gravity. The Schwarzschild-de Sitter black hole like solutions of $f(R)$ gravity were obtained for a positively constant and a non-constant curvature scalar in [12] and [13], respectively. A black hole solution was obtained from $f(R)$ theories by requiring the negative constant curvature scalar [7]. If $1 + f'(R_0) > 0$, this black hole is similar to the Schwarzschild-AdS (SAdS) black hole. It was argued that $f(R)$ and SAdS black holes have no big difference in thermodynamic quantities when using the Euclidean action approach and replacing the Newtonian constant G by $G_{\rm eff} = G/(1 + f'(R_0))$ [7]. It is also interesting to study black hole solutions in $f(R)$ theory coupled to a matter field. In general, the field equations of $f(R)$ theory coupled to the matter field are very complicated and hence it is not easy to find exact analytical solutions. In order to construct the constant curvature scalar black hole solutions from $f(R)$ gravity coupled to the matter, the trace of its energy-momentum tensor $T_{\mu\nu}$ should be zero [14]. Two examples for the traceless $T_{\mu\nu}$ are Maxwell and Yang-Mills fields which were studied in [14, 15]. Thermodynamics and properties of these solutions were also studied in ample details [14]. It was found that these solutions are similar to the Reissner-Nordström–AdS (RNAdS) black hole when making appropriate replacements [14]. The Kerr-Newman black hole solutions with non-zero constant scalar curvature in $f(R)$-Maxwell theory, their thermodynamics, as well as their local and global stability were also studied in [16].

In this paper we would like to continue the investigation on the $f(R)$ black holes, by constructing a new class of charged rotating black string solutions in $R + f(R)$-Maxwell theory with constant curvature scalar. The traceless property of the energy-momentum tensor of the Maxwell field plays a crucial role in our derivation. With assumptions $R_0 < 0$ and $1+f'(R_0) > 0$ our solution is similar to charged black string solution in AdS space with suitable replacing the parameters. We will also suggest the suitable counterterm which removes the divergences of the action. We calculate the conserved and thermodynamic quantities of these black strings by using the counterterm method. We obtain a Smarr-type formula for the mass of the black string and check the validity of the first law of thermodynamics. We perform the stability analysis in the canonical ensemble and show that the black strings are always thermodynamically stable in $f(R)$-Maxwell theory with constant curvature scalar. Finally, we extend the study to the case where the Ricci scalar is not constant and in particular $R = R(r)$ and derive an analytical black string solution.

II. FIELD EQUATIONS AND SOLUTIONS

We start from the four-dimensional $R + f(R)$ theory coupled to the Maxwell field

$$I_G = -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(R + f(R) - F_{\mu\nu}F^{\mu\nu}\right) - \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\Theta(h), \tag{1}$$

where R is the Ricci scalar curvature, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and A_μ is the electromagnetic potential. The last term in Eq. (1) is the Gibbons-Hawking boundary term. It is required for the variational principle to be well-defined. The factor Θ represents the trace of the extrinsic curvature for the boundary $\partial\mathcal{M}$ and h is the induced metric on the boundary. The equations of motion can be obtained by varying the action (1) with respect to the gravitational field $g_{\mu\nu}$ and the gauge field A_μ which yields the following field equations

$$R_{\mu\nu}\left(1 + f'(R)\right) - \frac{1}{2}g_{\mu\nu}(R + f(R)) + \left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)f'(R) = 8\pi T_{\mu\nu}, \tag{2}$$

$$\nabla_\mu F^{\mu\nu} = 0, \tag{3}$$

with the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\eta}F_{\nu}{}^{\eta} - \frac{1}{4}g_{\mu\nu}F_{\lambda\eta}F^{\lambda\eta}\right). \tag{4}$$

The above energy-momentum tensor is traceless in four dimension, i. e., $T^{\mu}{}_{\mu} = 0$. As we mentioned already this property plays an important role in our derivation. In Eq. (2) the "prime" denotes differentiation with respect to curvature scalar R. Assuming the constant curvature scalar $R = R_0$, the trace of Eq. (2) yields

$$R_0\left(1 + f'(R_0)\right) - 2\left(R_0 + f(R_0)\right) = 0, \tag{5}$$

Solving the above equation for negative R_0, gives

$$R_0 = \frac{2f(R_0)}{f'(R_0) - 1} \equiv 4\Lambda_{\rm f} < 0. \tag{6}$$

Substituting the above relation into Eq. (2), we obtain the following equation for Ricci tensor

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\left(\frac{f(R_0)}{f'(R_0) - 1}\right) + \frac{2}{1 + f'(R_0)}T_{\mu\nu}. \tag{7}$$

Now, we want to construct charged rotating black string solutions of the field equations (2) and (3) and investigate their properties. We are looking for the four-dimensional rotating solution with cylindrical or toroidal horizons. The metric which describes such a spacetime can be written in the following form [17, 18]

$$\begin{aligned} ds^2 &= -N(r)\left(\Xi dt - a d\phi\right)^2 + r^2\left(\frac{a}{l^2}dt - \Xi d\phi\right)^2 + \frac{dr^2}{N(r)} + \frac{r^2}{l^2}dz^2, \\ \Xi^2 &= 1 + \frac{a^2}{l^2}, \end{aligned} \tag{8}$$

where a is the rotation parameter. The function $N(r)$ should be determined and l has the dimension of length which is related to the constant $\Lambda_{\rm f}$ by the relation $l^2 = -3/\Lambda_{\rm f}$. The two dimensional space, t=constant and r =constant, can be (i) the flat torus model T^2 with topology $S^1 \times S^1$, and $0 \le \phi < 2\pi$, $0 \le z < 2\pi l$, (ii) the standard cylindrical model with topology $R \times S^1$, and $0 \le \phi < 2\pi$, $-\infty < z < \infty$, and (iii) the infinite plane R^2 with $-\infty < \phi < \infty$ and $-\infty < z < \infty$. We will focus upon (i) and (ii). The Maxwell equation (3) can be integrated immediately to give

$$\begin{aligned} F_{tr} &= \frac{q\Xi}{r^2}, \\ F_{\phi r} &= -\frac{a}{\Xi}F_{tr}, \end{aligned} \tag{9}$$

where q is the charge parameter of the black string. Substituting the Maxwell fields (9) as well as the metric (8) in the field equation (2) with constant curvature, the non-vanishing independent components of the field equations for $a = 0$ reduce to

$$(1 + f'(R_0))\left(2r^4\frac{d^2N(r)}{dr^2} + 4r^3\frac{dN(r)}{dr} + R_0 r^4\right) - 4q^2 = 0, \tag{10}$$

$$(1 + f'(R_0))\left(4r^3\frac{dN(r)}{dr} + 4r^2N(r) + R_0 r^4\right) + 4q^2 = 0. \tag{11}$$

One can easily show that the above equations have the following solution

$$N(r) = -\frac{2m}{r} + \frac{q^2}{(1 + f'(R_0))r^2} - \frac{R_0}{12}r^2, \tag{12}$$

where m is an integration constant which is related to the mass of the string. One can also check that these solutions satisfy equations (2)-(3) in the rotating case where $a \neq 0$. It is apparent that this spacetime is similar with asymptotically AdS black string. Indeed, with the following replacement

$$\frac{q^2}{(1 + f'(R_0))} \to Q^2 \tag{13}$$

$$\frac{R_0}{4} \to \Lambda \tag{14}$$

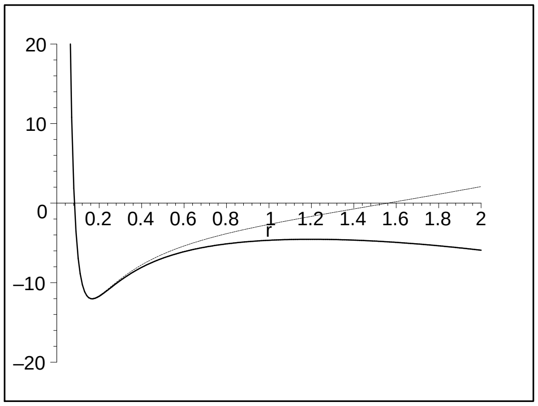

FIG. 1: The function $N(r)$ versus r for $m = 2$, $f'(R_0) = 2$ and $q = 1$. $R_0 = 12$ (bold line) and $R_0 = -12$ (continuous line).

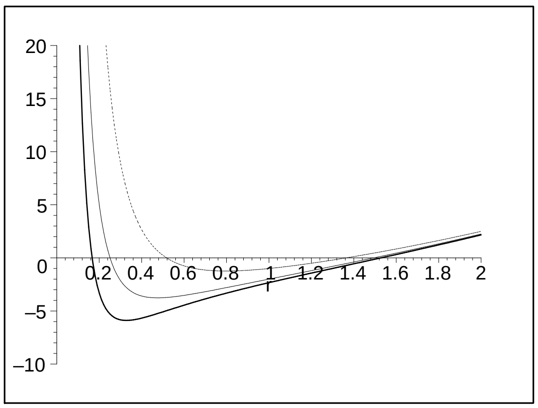

FIG. 2: The function $N(r)$ versus r for $m = 2$, $q = 1$ and $R_0 = -12$. $f'(R_0) = 0.5$ (bold line), $f'(R_0) = 0$ (continuous line) and $f'(R_0) = -0.5$ (dashed line).

the solution reduces to the asymptotically AdS charged black string for $\Lambda = -3/l^2$ [17]. Next we study the physical properties of the solutions. The Kretschmann scalar for this solution is given by

$$R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} = \frac{8}{3r^8(1+f'(R_0))^2}\left[r^2(\frac{1}{16}{R_0}^2r^6 + 18m^2)(1+f'(R_0))^2 - 36mrq^2(1+f'(R_0)) + 21q^4\right]. \tag{15}$$

When $r \to 0$, the dominant term in the Kretschmann scalar is $56q^4/[(1+f'(R_0))^2r^8]$. Therefore we have an essential singularity located at $r = 0$. The Kretschmann scalar also approaches ${R_0}^2/6$ as $r \to \infty$. As one can see from Eq.

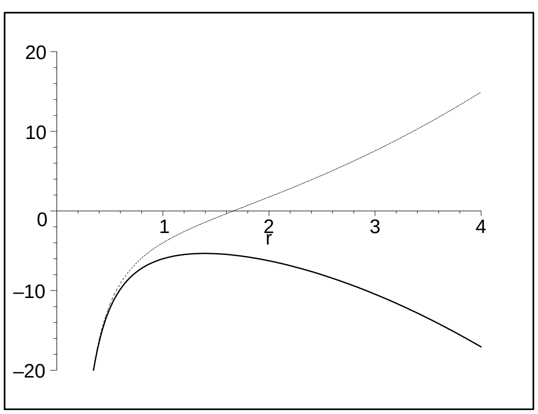

FIG. 3: The function $N(r)$ versus r for $m = 2$, $f'(R_0) = -2$ and $q = 1$. $R_0 = 12$ (bold line) and $R_0 = -12$ (continuous line).

(12), the solution is ill-defined for $f'(R_0) = -1$. The cases with $f'(R_0) > -1$ and $f'(R_0) < -1$ should be considered separately. In the first case where $f'(R_0) > -1$, there exist a cosmological horizon for $R_0 > 0$, while there is no cosmological horizons if $R_0 < 0$ (see fig. 1). Indeed, for $1 + f'(R_0) > 0$ and $R_0 < 0$ the black string can have two inner and outer horizons provided the parameters of the solutions are chosen suitably (see fig. 2). In the latter case ($f'(R_0) < -1$), the signature of the spacetime changes and the conserved quantities such as mass and angular momenta become negative, as we will see in the next section, thus this is not a physical case and we rule it out from our consideration (see fig. 3).

III. CONSERVED AND THERMODYNAMIC QUANTITIES

Next, we calculate the conserved quantities of the solutions by using the counterterm method inspired by (A)dS/CFT correspondence [19]. The spacetimes under consideration in this paper has zero curvature boundary, $R_{abcd}(h) = 0$, and therefore the counterterm for the stress energy tensor should be proportional to h^{ab}. We find the suitable counterterm which removes the divergences of the action in the form

$$I_{\rm ct} = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\sqrt{-\frac{R_0}{3}}, \tag{16}$$

where $R_0 < 0$. Having the total finite action $I = I_G + I_{\rm ct}$ at hand, one can use the quasilocal definition to construct a divergence free stress-energy tensor [20]. Thus the finite stress-energy tensor can be written as

$$T^{ab} = \frac{1}{8\pi}\left[\Theta^{ab} - \Theta h^{ab} - \sqrt{-\frac{R_0}{3}}h^{ab}\right]. \tag{17}$$

The first two terms in Eq. (17) are the variation of the action (1) with respect to h_{ab}, and the last term is the variation of the boundary counterterm (16) with respect to h_{ab}. To compute the conserved charges of the spacetime, one should choose a spacelike surface $\mathcal{B}$ in $\partial\mathcal{M}$ with metric σ_{ij}, and write the boundary metric in ADM (Arnowitt-Deser-Misner) form:

$$h_{ab}dx^a dx^a = -N^2dt^2 + \sigma_{ij}\left(d\varphi^i + V^i dt\right)\left(d\varphi^j + V^j dt\right),$$

where the coordinates φ^i are the angular variables parameterizing the hypersurface of constant r around the origin, and N and V^i are the lapse and shift functions respectively. When there is a Killing vector field ξ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (17) can be written as

$$Q(\xi) = \int_{\mathcal{B}} d^2x\sqrt{\sigma}T_{ab}n^a\xi^b, \tag{18}$$

where σ is the determinant of the metric σ_{ij}, ξ and n^a are, respectively, the Killing vector field and the unit normal vector on the boundary $\mathcal{B}$. The first Killing vector of the spacetime is $\xi = \partial/\partial t$, and therefore its associated conserved charge of the string is the mass per unit volume. A simple calculation gives

$$M \;=\; \int_{\mathcal{B}} d^2x\sqrt{\sigma}T_{ab}n^a\xi^b = \frac{(3\Xi^2 - 1)m}{8\pi l}\left[1 + f'(R_0)\right]. \tag{19}$$

The second conserved quantity is the angular momentum per unit volume associated with the rotational Killing vectors $\varsigma = \partial/\partial\phi$ which can be calculated as

$$J = \int_{\mathcal{B}} d^2x\sqrt{\sigma}T_{ab}n^a\varsigma^b = \frac{3\Xi m\sqrt{\Xi^2 - 1}}{8\pi}\left[1 + f'(R_0)\right]. \tag{20}$$

For $a = 0$ ($\Xi = 1$), the angular momentum per unit volume vanishes, and therefore a is the rotational parameters of the spacetime. Next we calculate the entropy of the black string. Let us first give a brief discussion regarding the entropy of the black hole solutions in $f(R)$ gravity. To this aim, we follow the arguments presented in [21]. If one use the Noether charge method for evaluating the entropy associated with black hole solutions in $f(R)$ theory with constant curvature, one finds [12]

$$S = \frac{A}{4G}f'(R_0), \tag{21}$$

where $A = 4\pi r_+^2$ is the horizon area. As a result, in $f(R)$ gravity, the entropy does not obey the area law and one obtains a modification of the " area law". Motivated by the above argument, for the rotating black string solution in $R + f(R)$ gravity, we find the entropy per unit length of the string as

$$S = \frac{r_+^2 \Xi}{4l} \left[1 + f'(R_0)\right]. \tag{22}$$

Then we obtain the temperature and angular velocity of the horizon by analytic continuation of the metric. Although our solution is not static, the Killing vector

$$\chi = \partial_t + \Omega \partial_\phi \tag{23}$$

is the null generator of the event horizon where Ω is the angular velocity of the outer horizon. The analytical continuation of the Lorentzian metric by $t \to i\tau$ and $a \to ia$ yields the Euclidean section, whose regularity at $r = r_+$ requires that we should identify $\tau \sim \tau + \beta_+$ and $\phi \sim \phi + i\beta_+\Omega_+$ where β_+ and Ω_+ are the inverse Hawking temperature and the angular velocity of the horizon. We find

$$T = \frac{1}{4\pi\Xi}\left(\frac{dN(r)}{dr}\right)_{r=r_+} = -\frac{\left[R_0 r_+^4 (1 + f'(R_0)) + 4q^2\right]}{16\pi\Xi[1 + f'(R_0)]r_+^3}, \tag{24}$$

$$\Omega = \frac{a}{\Xi l^2}, \tag{25}$$

where we have used equation $N(r_+) = 0$ for omitting the mass parameter m from temperature expression. Since $1 + f'(R_0) > 0$, therefore the temperature is non negative provided

$$R_0 r_+^4 (1 + f'(R_0)) \leq -4q^2 \to R_0 \leq -\frac{4q^2}{r_+^4 (1 + f'(R_0))}, \tag{26}$$

where the equality holds for extremal black string with zero temperature. The next quantity we are going to calculate is the electric charge of the string. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurface. The normal vectors to such hypersurface are

$$u^0 = \frac{1}{N}, \quad u^r = 0, \quad u^i = -\frac{V^i}{N}, \tag{27}$$

where N and V^i are the lapse function and shift vector. Then the electric field is $E^\mu = g^{\mu\rho} F_{\rho\nu} u^\nu$, and the electric charge per unit length of the string can be found by calculating the flux of the electric field at infinity,

$$Q = \frac{\Xi q}{4\pi l \sqrt{1 + f'(R_0)}}. \tag{28}$$

The electric potential U, measured at infinity with respect to the horizon, is defined by [22]

$$U = A_\mu \chi^\mu \left|_{r\to\infty} - A_\mu \chi^\mu \right|_{r=r_+}, \tag{29}$$

where χ is the null generator of the event horizon given in Eq. (23). One can easily obtain the electric potential as

$$U = \frac{q}{\Xi r_+}\sqrt{1 + f'(R_0)}. \tag{30}$$

Then, we consider the first law of thermodynamics for the black string. In order to do this, we obtain the mass M as a function of extensive quantities S, J and Q. Using the expression for the mass, the angular momenta, the entropy and the charge given in Eqs. (19), (20), (22) and (28) and the fact that $N(r_+) = 0$, one can obtain a Smarr-type formula as

$$M(S, J, Q) = \frac{J(3Z - 1)}{3l\sqrt{Z(Z - 1)}}, \tag{31}$$

where $Z = \Xi^2$ is the positive real root of the following equation:

$$\frac{3\sqrt{Z-1}\pi^2 Q^2 l^2 [1 + f'(R_0)]^2 - 2J\pi\sqrt{\sqrt{Z}(1 + f'(R_0))Sl} + 3S^2\sqrt{Z-1}}{2\pi\sqrt{\sqrt{Z}(1 + f'(R_0))Sl}} = 0. \tag{32}$$

One may then regard the parameters S, J and Q as a complete set of extensive parameters for the mass $M(S,J,Q)$ and define the intensive parameters conjugate to S, J and Q. These quantities are the temperature, the angular velocities and the electric potential

$$T = \left(\frac{\partial M}{\partial S}\right)_{J,Q} = -\frac{J\left\{\left[\pi Ql(1+f'(R_0))\right]^2 - 3S^2\right\}}{3Sl\sqrt{Z(Z-1)}\left\{\left[\pi Ql(1+f'(R_0))\right]^2 + S^2\right\}}, \tag{33}$$

$$\begin{aligned} \Omega &= \left(\frac{\partial M}{\partial J}\right)_{S,Q} \\ &= \frac{3\,(3Z-1)\left\{\left[\pi Ql(1+f'(R_0))\right]^2 + S^2\right\} - 4\pi J\sqrt{\sqrt{Z}(Z-1)(1+f'(R_0))Sl}}{9l\sqrt{Z(Z-1)}\left\{\left[\pi Ql(1+f'(R_0))\right]^2 + S^2\right\}}, \end{aligned} \tag{34}$$

$$U = \left(\frac{\partial M}{\partial Q}\right)_{S,J} = \frac{4\pi^2 QlJ[1+f'(R_0)]^2}{3\sqrt{Z(Z-1)}\left\{\left[\pi Ql(1+f'(R_0))\right]^2 + S^2\right\}}. \tag{35}$$

Numerical calculations show that the intensive quantities calculated by Eqs. (33)-(35) coincide with Eqs. (24), (25) and (30), respectively. Thus, these thermodynamics quantities satisfy the first law of thermodynamics

$$dM = TdS + \Omega dJ + UdQ. \tag{36}$$

IV. THERMAL STABILITY OF BLACK STRING

Finally, we investigate the thermal stability of rotating black string solutions in $f(R)$ gravity coupled to a matter field. The stability of a thermodynamic system with respect to small variations of the thermodynamic coordinates is usually performed by analyzing the behavior of the entropy $S(M,J,Q)$ around the equilibrium. The local stability in any ensemble requires that $S(M,J,Q)$ be a convex function of the extensive variables or its Legendre transformation must be a concave function of the intensive variables. The stability can also be studied by the behavior of the energy $M(S,J,Q)$ which should be a convex function of its extensive variables. Thus, the local stability can in principle be carried out by finding the determinant of the Hessian matrix of $M(S,J,Q)$ with respect to its extensive variables X_i, $\mathbf{H}^M_{X_iX_j} = [\partial^2 M/\partial X_i \partial X_j]$ [22, 23]. In our case the mass M is a function of entropy, angular momenta, and charge. The number of thermodynamic variables depends on the ensemble that is used. In the canonical ensemble, the charge and the angular momenta are fixed parameters, and therefore the positivity of the $(\partial^2 M/\partial S^2)_{J,Q}$ is sufficient to ensure local stability. We find that the black string solutions are always thermally stable independent of the value of the parameters q and Ξ. We have shown the behavior of $(\partial^2 M/\partial S^2)_{J,Q}$ as a function q and r_+ for different value of Ξ and $f'(R_0)$ in figures 4-6. These figures show that the black string solutions in $f(R)$-Maxwell theory with constant curvature scalar are always thermally stable.

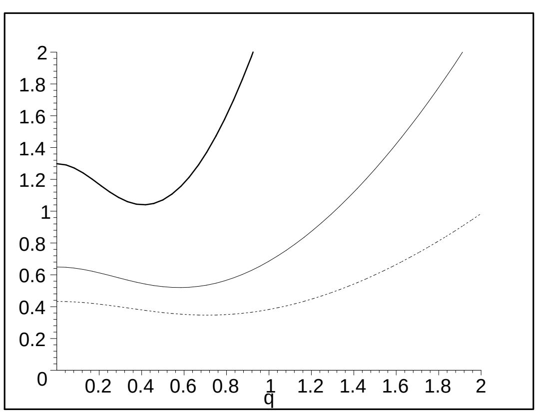

FIG. 4: The function $(\partial^2 M/\partial S^2)_{J,Q}$ versus q for $l = 1$, $\Xi = 1.25$, $r_+ = 0.7$ and $R_0 = -12$. $f'(R_0) = 0$ (bold line), $f'(R_0) = 1$ (continuous line) and $f'(R_0) = 2$ (dashed line).

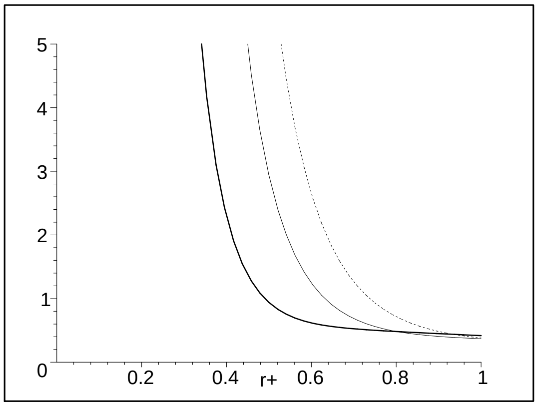

FIG. 5: The function $(\partial^2 M/\partial S^2)_{J,Q}$ versus r_+ for $l = 1$, $\Xi = 1.25$, $R_0 = -12$ and $f'(R_0) = 1$. $q = 0.5$ (bold line), $q = 1$ (continuous line) and $q = 1.5$ (dashed line).

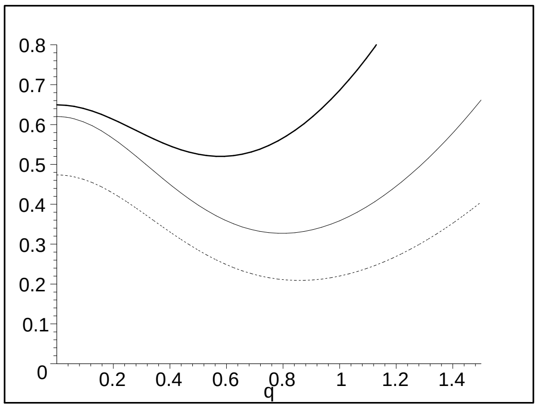

FIG. 6: The function $(\partial^2 M/\partial S^2)_{J,Q}$ versus q for $l = 1$, $f'(R_0) = 1$ and $R_0 = -12$. $\Xi = 1.25$, (bold line), $\Xi = 1.75$, (continuous line) and $\Xi = 2.25$, (dashed line).

V. SOLUTION WITH NON CONSTANT RICCI SCALAR

In this section we would like to extend the study to the case where the Ricci scalar is not a constant, instead we reconstruct it as $R = R(r)$ as a result of our calculations. In this case, we find out that we can obtain solution only in the absence of the matter field. As we mentioned in the introduction, in general, the field equations of $f(R)$ theory coupled to the matter field are very complicated and hence it is not easy to find exact analytical solutions. Thus we only consider the uncharged black string solution. We start with the following action

$$S = \frac{1}{16\pi} \int d^4x\sqrt{-g} f(R)\,, \tag{37}$$

where $f(R)$ is a generic function of the Ricci scalar R. We also modify our metric (8) a bit as follow

$$ds^2 \; = \; -N(r)\mathrm{e}^{2\alpha(r)} \left(\Xi dt - a d\phi\right)^2 + r^2 \left(\frac{a}{l^2}dt - \Xi d\phi\right)^2 + \frac{dr^2}{N(r)} + \frac{r^2}{l^2}dz^2, \tag{38}$$

where we have added an additional function $\alpha(r)$ in the metric coefficients. For simplicity we only consider the non-rotating black string with $a = 0$, thus the above metric reduces to

$$ds^2 = -N(r)\mathrm{e}^{2\alpha(r)}dt^2 + \frac{dr^2}{N(r)} + r^2 d\phi^2 + \frac{r^2}{l^2}dz^2. \tag{39}$$

The scalar curvature for metric (39) reads

$$R \; = \; -3\,\frac{dN\,(r)}{dr}\frac{d\alpha\,(r)}{dr} - 2\,N\,(r)\left[\frac{d}{dr}\alpha\,(r)\right]^2 - \frac{d^2N\,(r)}{dr^2} - 2\,N\,(r)\,\frac{d^2\alpha\,(r)}{dr^2}$$

$$-\frac{4}{r}\frac{dN\left(r\right)}{dr}-\frac{4N\left(r\right)}{r}\frac{d\alpha\left(r\right)}{dr}-\frac{2N\left(r\right)}{r^{2}}\,. \tag{40}$$

We use the Lagrangian multipliers method. In the framework of Friedmann-Robertson-Walker universe this method was studied in [25–27], while for static spherically symmetric black hole solutions it was investigated in [13, 28]. In this approach one may consider the scalar curvature R as independent Lagrangian coordinates in addition to the functions $\alpha(r)$ and $N(r)$, which appear from the metric line element.

Introducing the Lagrangian multipliers λ, after using (40), the action (37) can be written

$$\begin{aligned} S &\equiv \frac{1}{16\pi}\int dt\int dr\left(e^{\alpha(r)}r^{2}\right)\left\{f(R)-\lambda\left\{R+\left\{3\left[\frac{d}{dr}N\left(r\right)\right]\frac{d}{dr}\alpha\left(r\right)\right.\right.\right. \\ &+2\,N\left(r\right)\left[\frac{d}{dr}\alpha\left(r\right)\right]^{2}+\frac{d^{2}N\left(r\right)}{dr^{2}}+2\,N\left(r\right)\frac{d^{2}\alpha\left(r\right)}{dr^{2}}+\frac{4}{r}\frac{dN\left(r\right)}{dr} \\ &+\frac{4N\left(r\right)}{r}\frac{d\alpha\left(r\right)}{dr}+\frac{2N\left(r\right)}{r^{2}}\quad\}\}\}\,. \end{aligned} \tag{41}$$

Varying the above action with respect to R, one gets

$$\lambda=f'(R), \tag{42}$$

where the prime denotes the derivative with respect to the scalar curvature R. Substituting this value and integrating by part, the Lagrangian takes the form

$$\begin{aligned} L(\alpha,d\alpha/dr,N,dN/dr,R,dR/dr) &= e^{\alpha}\left\{r^{2}\left[f(R)-Rf'(R)\right]-2f'(R)\left(r\frac{dN(r)}{dr}+N(r)\right)\right. \\ &\left.+f''(R)\frac{dR}{dr}r^{2}\left(\frac{dN(r)}{dr}+2N(r)\frac{d\alpha(r)}{dr}\right)\right\}\,. \end{aligned} \tag{43}$$

Making the variation with respect to α, one gets the first equation of motion

$$\begin{aligned} &\frac{Rf'(R)-f(R)}{f'(R)}+\frac{2}{r^{2}}\left[N(r)+r\frac{dN(r)}{dr}\right] \\ &+\frac{2N(r)f''(R)}{f'(R)}\left[\frac{d^{2}R}{dr^{2}}+\left(\frac{dN(r)/dr}{2N(r)}\right)\frac{dR}{dr}+\frac{f'''(R)}{f''(R)}\left(\frac{dR}{dr}\right)^{2}\right]=0\,. \end{aligned} \tag{44}$$

The variation with respect to $N(r)$ leads the second equation of motion

$$\left[\frac{d\alpha(r)}{dr}\left(\frac{f''(R)}{f'(R)}\frac{dR}{dr}\right)-\frac{f''(R)}{f'(R)}\frac{d^{2}R}{dr^{2}}-\frac{f'''(R)}{f'(R)}\left(\frac{dR}{dr}\right)^{2}\right]=0\,, \tag{45}$$

while by making the variation with respect to R, we recover Eq. (40). Given $f(R)$, together with equation (40), the above equations form a system of three differential equations for the three unknown quantities $\alpha(r), N(r)$ and $R(r)$. We would like to note that one advantage of this approach is that α does not appear in Eq.(44). In what follow, we will find exact solutions of the above system of differential equations.

In the special case of constant curvature $R=R_0$ and $\alpha=$ constant, it is easy to show that the only solution of Eqs. (40) and(44) is the Schwarzshild de Sitter black string solution with flat horizon,

$$N(r)=-\frac{2m}{r}-\frac{\Lambda}{3}r^{2}, \tag{46}$$

where m is a constant of integration which can be interpreted as the mass parameter of the black string and we have defined $2\Lambda\equiv R_0-f(R_0)/f'(R_0)$ and $R_0=4\Lambda$. Notice that in the absence of the matter field, $q=0$, solution (12) coincides with the result obtained in (46), as expected.

Next, we consider the case of non constant Ricci curvature, but still with $\alpha=$ constant. From Eq. (45) we have

$$f'''\left(\frac{dR}{dr}\right)^{2}+f''\left(\frac{d^{2}R}{dr^{2}}\right)=\frac{d^{2}}{dr^{2}}f'(R)=0, \tag{47}$$

which has the following solution,

$$f'(R) = mr + n, \tag{48}$$

where m and n are two integration constants. Given the explicit form of R, we may find r as a function of Ricci scalar and reconstruct $f'(R)$ realizing such solution. From Eq. (40) with constant α, one gets

$$R = -\frac{d^2N(r)}{dr^2} - \frac{4}{r}\frac{dN(r)}{dr} - \frac{2N(r)}{r^2}. \tag{49}$$

Using the fact that $(f''(R))dR/dr = df'(R)/dr = m$ and $df(R)/dr = f'(R)dR/dr$, and multiplying Eq. (44) by $f'(R)$, we arrive at

$$-\frac{d^2N(r)}{dr^2}\left(m + \frac{n}{r}\right) + \frac{4mN(r)}{r^2} + \frac{2nN(r)}{r^3} - \frac{m}{r}\frac{dN(r)}{dr} = 0\,. \tag{50}$$

When $m = 0$, the solution of the above equation is ones obtained in (46). For $n = 0$, the general solution is

$$N(r) = -C_1r^2 + \frac{C_2}{r^2}. \tag{51}$$

Substituting in Eq. (49) we again arrive at constant Ricci scalar, $R = 12C_1$. Although in this case $f'(R) = df(R)/dR = mr$ is not a constant, but still we have $df(R)/dr = 0$, which implies that $f(R) = \text{constant}$. Next we look for the most general solution of Eq.(50) with $n \neq 0$ and $m \neq 0$. Solving (50), we find

$$N(r) = -C_1r^2 + \frac{C_2}{r}\left[2\,n^3 - 3mn^2r + 6\,m^2r^2n - 6m^3r^3\ln\left(m + \frac{n}{r}\right)\right], \tag{52}$$

where C_1 and C_2 are two arbitrary constants. Given solution (52) one can basically construct $f(R)$ by using Eqs. (48) and (49). In order to simplify the above solution, we choose $n = 1$ and $C_2 = -1/m$,

$$N(r) = 3 - \frac{2}{mr} - 6\,mr - C_1r^2 + 6m^2r^2\ln\left(m + \frac{1}{r}\right). \tag{53}$$

For this general case the Ricci scalar becomes

$$R(r) = 12C_1 - 72m^2\ln\left(m + \frac{1}{r}\right) + \frac{6\left(12m^3r^3 + 18m^2r^2 + 4mr - 1\right)}{r^2(mr+1)^2}. \tag{54}$$

which is clearly not a constant. Now we want to reconstruct the corresponding $f(R)$ theory. From Eq.(48), for $n = 1$ one has

$$f'(R) = \frac{df(R)}{dR} = \frac{df(R)}{dr}\frac{dr}{dR} = mr + 1. \tag{55}$$

Integrating (55), by using (54), we get

$$f[R(r)] = -36m^2\ln\left(m + \frac{1}{r}\right) + \frac{3\left(12m^3r^3 + 18m^2r^2 + 4mr - 2\right)}{r^2(ar+1)^2}. \tag{56}$$

Combining Eqs. (54), (55) and (56), one gets the following differential equation for function $f(R)$,

$$\frac{3m^2}{[f'(R)]^2[f'(R)-1]^2} + f(R) - \frac{R}{2} + 6C_1 = 0. \tag{57}$$

This equation has a simple solution as

$$f(R) = \frac{R}{2} - 6C_1 - 48m^2, \tag{58}$$

but it has also another complicated solution which we have not presented it here. Thus we have found a black string solution in $f(R)$ gravity with non constant Ricci scalar. This approach also leads to construct Ricci scalar as a function of r, as given in Eq. (54). The obtained solutions in this section differ from that presented in [28] for axially symmetric solutions in $f(R)$ gravity. It is worth mentioning that metric (38) has a good property for which its static and rotating solutions coincide and so in this section we only study the static case. Following the approach of this section, one can easily check that solution (52) can be deduced for rotating case where $a \neq 0$. Besides, in this section we only considered the case with $\alpha = \text{constant}$, and derived the metric function (52) as well as $R(r)$. The study can also be generalized to the case where $\alpha = \alpha(r)$. We leave it and also thermodynamic considerations of the obtained solution in this section, for future investigations.

VI. CONCLUSIONS

In order to obtain the constant curvature black hole solution in $f(R)$ gravity theory coupled to a matter field, the trace of the energy-momentum tensor of the matter field should be zero [14]. Since the energy-momentum tensor of Maxwell field is traceless in four dimensions, therefore spherically symmetric black hole solutions from $f(R)$ theory coupled to Maxwell field was derived in four dimensional spacetime [14].

In this paper we continued the study by constructing a new class of charged rotating solutions in $f(R)$-Maxwell theory with constant curvature scalar. This class of solutions describe the four dimensional charged rotation black string with cylindrical or toroidal horizons with zero curvature boundary. These solutions are similar to asymptotically AdS black string of Einstein-Maxwell gravity with suitably replacement of the parameters. However, the solution presented in this paper has at least two differences from AdS black string solutions of Einstein-Maxwell gravity. First, the conserved and thermodynamic quantities computed here depend on function $f'(R_0)$ and differ completely from those of Einstein theory in AdS spaces. Clearly the presence of the general function $f'(R_0)$ changes the physical values of conserved and thermodynamic quantities. Second, unlike Einstein gravity, the entropy does not obey the area law for black string solutions in $f(R)$-Maxwell theory as one can see from Eq (22). We studied the physical properties of the solutions and found a suitable conterterm which removes the divergence of the action. We obtained mass and angular momenta of the string through the use of conterterm method. We also derived the entropy of the black string in $f(R)$ gravity which has a modification from the area law. We obtained a Smarr-type formula for the mass namely $M(S, J, Q)$ and checked that the obtained conserved and thermodynamic quantities satisfy the first law of black hole thermodynamics. Finally, we explored the thermal stability of the solutions in the canonical ensemble and showed that the black strings derived from $f(R)$- Maxwell theory are always thermally stable. This is commensurate with the fact that there is no Hawking-Page phase transition for black objects with zero curvature horizon [24].

We also extend the study to the case where the Ricci scalar is not a constant. For this purpose we used the Lagrangian multipliers method and found an exact black string solution in $f(R)$ gravity. In this approach one may consider the scalar curvature $R(r)$ as an independent Lagrangian coordinates in addition to the metric functions and deduce $R(r)$ as a solution of the field equations. We found the explicit form of $R(r)$ as well as the metric function which has a logaritmic term. It is worth noting that since for the non constant Ricci scalar, the field equations of $f(R)$ theory coupled to the matter field become very complicated, in this case, we could only derived analytical solution in the absence of the matter field.

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